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A note on the description of an attached supersonic boundary layer near a point of injection cut-off

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Supersonic boundary-layer flow on a flat plate near a point of injection cut-off is considered. The goal is to develop a non-singular solution in the limit of large Reynolds number in the vicinity of the boundary-condition change. An application of Goldstein-type singular solutions shows that an interaction-type theory is required even for transverse velocity jumps on the boundary $O(R^{-\frac{1}{2}})$. The interaction analysis is developed in terms of the linearized triple-deck theory described by Smith & Stewartson (1973*a*). The analytically derived solution provides a continuous pressure and wall-shear distribution.

1. Introduction

Boundary-layer flows involving an injection cut-off point fall within the class of problems characterized by discontinuous boundary conditions at one or more points of the flow. For such problems, classical boundary-layer theory is not valid in the neighbourhood of these discontinuities. Rather, a local interaction theory must be used in order to describe the flow in such a neighbourhood. The purpose of this note is to apply an existing local interaction theory (Smith & Stewartson 1973*a*) to the injection cut-off problem.

To this end, we consider a supersonic uniform flow parallel to a hot (cold) semi-infinite flat plate. A gas is injected at the wall with a mass rate distribution $\dot{m}$ normal to the plate described by

$$\dot{m} = C(2xR)^{-\frac{1}{2}}[1-H(x-1)], \tag{1.1}$$

where H is the Heaviside step function. The variable x is measured along the plate starting from the leading edge and has been non-dimensionalized with respect to the finite length of injection L.

The Reynolds number of the flow is defined by $R = U_\infty L/\nu_\infty \gg 1$, where U_∞ and ν_∞ are the values of the uniform velocity and the kinematic viscosity of the external flow respectively. The injection parameter C is to be specified and is chosen such that $C < C_0$, where C_0 is a critical value beyond which the boundary layer separates. Some critical values for various flow conditions are tabulated in Amr & Kassoy (1973).

Classical boundary-layer theory remains valid upstream of $x = 1$ and no separation occurs at any point in the flow. Our emphasis will be on the structure of the flow in the neighbourhood of the cut-off point $x = 1$. To this end, we first describe the flow for $\xi = x - 1 \to 0+$. Owing to the sudden change in the boundary conditions at the wall, the flow variables and/or their derivatives at the cut-off point will not be continuous functions. (The effects of such discontinuities were observed by Wolfram & Walker (1970) in a study of the influence of upstream injection on heat transfer in a supersonic boundary layer.) Thus a thin layer located around $x = 1$ will then be constructed to provide a smooth transition as the flow variables change from their forms at $x = 1-$ to those at $x = 1+$. In this region classical boundary-layer theory is not a valid model of the flow. Rather, a local interaction theory (triple-deck theory) must be used. This theory has been developed by Stewartson & Williams (1969), Stewartson (1969) and Messiter (1970). This approach was used by Stewartson & Williams (1969) for studying interactions between shock waves and boundary layers, by Stewartson (1969), Brown & Stewartson (1970) and Messiter (1970) for trailing-edge studies, by Smith & Stewartson (1973*a*, *b*) for strong-injection turn-on studies, by Smith (1973) for the study of the influence of small bumps on boundary-layer flows, by Stewartson (1970*a*, *b*) for convex- and concave-corner flows, and finally, by Messiter & Hu (1975) for wall-curvature discontinuities. The linearized form of the triple-deck theory has been described by Smith (1972) and Stewartson & Smith (1973*a*) in the context of the slot-injection problem. The analysis therein is used to describe the disturbance to an upstream Blasius boundary layer caused by initiation of wall injection of magnitude $\dot{m} = O(V_W R^{-\frac{3}{8}})$ over a streamwise distance $O(R^{-\frac{3}{8}})$ when $V_W \ll 1$. For these conditions the lower-deck system has a linear form which can be solved analytically.

In this note we consider the disturbance to the upstream boundary layer associated with (1.1) for $x < 1$ caused by the abrupt cut-off of injection at $x = 1$ It is shown that the linear lower-deck system for this problem can be reduced to that given by Smith & Stewartson (1973*a*). Hence their results may be used directly in the present problem. Of course, the physical interpretation of those results differs in the present work because the boundary conditions are altered.

2. Flow downstream of the cut-off point

We consider the flow in a boundary layer adjacent to a semi-infinite flat plate. A gas is being injected (weak injection) over a region starting from the leading edge of the plate and extending a finite distance along the plate to $x = 1$ (the cut-off location). The mass rate distribution is given by (1.1), where for simplicity the gas injected at the surface will be taken to be the same as the gas in the external flow.

The analysis of the flow in the boundary layer both upstream and downstream of the cut-off point is best carried out in terms of the Howarth–Dorodnitsyn co-ordinates defined by

$$\overline{Y} = \int_0^Y \rho \, dY, \quad Y = yR^{\frac{1}{2}}, \tag{2.1}$$

$$U = \Psi_{\overline{Y}}, \quad V = -\rho^{-1}[\Psi_x + U(\partial \overline{Y}/\partial x)_Y], \qquad (2.2), (2.3)$$

where y is measured normal to the plate and has been non-dimensionalized with respect to L. The transformed stream function Ψ is related to the physical stream function Ψ^* by the relation $\Psi = (\Psi^*/U_\infty L)R^{\frac{1}{2}}$. Furthermore, we use the Chapman viscosity law $\rho\mu = \text{constant} = G$, where μ is the viscosity of the flow non-dimensionalized with respect to the external-flow value.† The basic equations in the boundary layer are obtained from the full Navier–Stokes equation by using (2.1)–(2.3) and applying the limit process x, $\overline{Y}$ fixed as $R \to \infty$. In stream-function form we find

$$\Psi_{\overline{Y}}\Psi_{x\overline{Y}} - \Psi_x\Psi_{\overline{Y}\overline{Y}} = \Psi_{\overline{Y}\overline{Y}\overline{Y}}. \tag{2.4}$$

The boundary conditions are given by

$$\Psi_{\overline{Y}}(x, \overline{Y} \to \infty) = 1, \tag{2.5}$$

$$\Psi_{\overline{Y}}(x, 0) = 0, \quad \Psi_x(x, 0) = [-C/(2x)^{\frac{1}{2}}][1 - H(x-1)]. \qquad (2.6), (2.7)$$

For the purposes of the analysis of the transition layer at $x = 1$, we need only the basic properties of the flow just upstream of the cut-off point. To this end we note that for $0 < x < 1$ equations (2.4)–(2.7) admit a similarity solution in terms of the similarity variables defined by

$$\overline{\eta} = \overline{Y}/(2Gx)^{\frac{1}{2}}, \quad \Psi = (2Gx)^{\frac{1}{2}}f(\overline{\eta}).$$

In terms of these new variables, (2.4)–(2.7) reduce to the system of equations

$$f''' + ff'' = 0, \tag{2.8}$$

$$f(0) = -C/G^{\frac{1}{2}}, \quad f'(0) = 0, \quad f'(\overline{\eta} \to \infty) = 1. \tag{2.9}$$

The solutions of (2.8) and (2.9) can be extracted from the tables in Emmons & Leigh (1954). In particular, for $x = 1-$, $\Psi = \Psi_0(\overline{Y})$, where $\Psi_0(\overline{Y})$ satisfies

$$2G\Psi_0''' + \Psi_0\Psi_0'' = 0,$$

$$\Psi_0(0) = -2^{\frac{1}{2}}C, \quad \Psi_0'(0) = 0, \quad \Psi_0'(\overline{Y} \to \infty) = 1.$$

The properties of Ψ_0 as $\overline{Y} \to 0$ (or $Y \to 0$) are

$$\Psi_0(\overline{Y} \to 0) = -2^{\frac{1}{2}}C + \frac{a}{G^{\frac{1}{2}}}\frac{\overline{Y}^2}{2} + \frac{Ca}{2^{\frac{1}{2}}G^{\frac{3}{2}}}\frac{\overline{Y}^3}{6} + \ldots, \tag{2.10}$$

$$\Psi_0(Y \to 0) = -2^{\frac{1}{2}}C + \frac{a}{G^{\frac{1}{2}}T_w}\frac{Y^2}{2} + \frac{Ca}{2^{\frac{1}{2}}G^{\frac{3}{2}}T_W^2}\frac{Y^3}{6} + \ldots, \tag{2.11}$$

where T_w is the prescribed wall temperature. Here $a = f''(0)/\sqrt{2}$ and may be found from Emmons & Leigh's (1954) tables. It is to be noted that a depends on C and tends to zero as $C \to C_0$. Here C_0 is the critical injection value described earlier.

Now that we have established the basic properties of the flow upstream of the cut-off point, we turn our attention to the flow just downstream of the cut-off point. A Goldstein-type analysis (Goldstein 1930) of the singularity will be used. Thus for $\xi = x - 1 \to 0+$ we look for a solution of the form

$$\Psi(\xi, \overline{Y}) = \Psi_0(\overline{Y}) + \xi^m\Psi_1(\overline{Y}) + \ldots. \tag{2.12}$$

† This model leads to an uncoupling of the momentum equation from energy considerations which is valid as long as the pressure field is $o(1)$.

Such a form agrees with the initial profile $\Psi_0(\overline{Y})$ at $x = 1-$, but in general it does not satisfy the new boundary condition at the wall for $x = 1+$, namely $\Psi_x(x, 0) = 0$. Thus (2.12) represents a continuation of the flow in the upstream boundary layer but is not valid near the wall, where a different expansion must be specified as $\overline{Y} \to 0$. According to Goldstein's theory, this 'wall-layer' expansion takes the form

$$\Psi(\xi, \eta) = -2^{\frac{1}{2}} C + \xi^{n_1} F_0(\eta) + \xi^{n_2} F_1(\eta) + \dots, \tag{2.13}$$

where
$$\eta = \overline{Y}/\xi^{n_3}, \quad n_1 < n_2. \tag{2.14}$$

Following the standard reasoning (Goldstein 1930; Smith 1972) it may be shown that $m = n_1 = 2n_3 = \frac{2}{3}$. Similarly one finds

$$\Psi_1(\overline{Y}) = \frac{3^{\frac{1}{3}}\Gamma(\frac{1}{3})}{2^{\frac{3}{2}} a^{\frac{2}{3}} \Gamma(\frac{2}{3})} \Psi_0''(\overline{Y}), \quad F_0(\eta) = \frac{a}{2G^{\frac{1}{2}}} \eta^2. \qquad (2.15), (2.16)$$

The system describing $F_1(\eta)$,

$$GF_1''' - \frac{a}{G^{\frac{1}{2}}} \eta F_1' + \frac{a}{3} \frac{\eta^2}{G^{\frac{1}{2}}} F_1'' + \frac{a}{G^{\frac{1}{2}}} F_1 = 0, \tag{2.17}$$

$$F_1(0) = F_1'(0) = 0, \quad F_1'(\eta \to \infty) = \frac{Ca}{(2G)^{\frac{3}{2}}} \eta^2, \tag{2.18}$$

was found by Smith (1972) in a strong-injection initiation problem. In (2.15), Γ is the standard gamma function. The solution of (2.17) with (2.18) is given by

$$\begin{aligned} F_1(\eta) = \frac{Ca}{2^{\frac{1}{2}} G^{\frac{3}{2}}} \Bigg[\frac{\eta^3}{6} + \frac{1}{3} \frac{3^{\frac{2}{3}} G^{\frac{1}{2}}}{a^{\frac{1}{3}} \Gamma(\frac{2}{3})} \Bigg\{ \eta \int_0^\eta \exp\left(\frac{-a\eta_1^3}{9G^{\frac{3}{2}}}\right) d\eta_1 - \frac{2}{3} \int_0^\eta \eta_1 \exp\left(\frac{-a\eta_1^3}{9G^{\frac{3}{2}}}\right) d\eta_1 \\ + \frac{\eta^2}{3} \exp\left(\frac{-a\eta^3}{9G^{\frac{3}{2}}}\right) - \frac{a}{9G^{\frac{3}{2}}} \eta^3 \int_\eta^\infty \eta_1 \exp\left(\frac{-a\eta_1^3}{9G^{\frac{3}{2}}}\right) d\eta_1 \Bigg\} \Bigg]. \end{aligned} \tag{2.19}$$

Thus Ψ is given near $x = 1+$ by

$$\Psi(\xi, \overline{Y}) = \Psi_0(\overline{Y}) + \frac{3^{\frac{1}{3}}}{2^{\frac{3}{2}} a^{\frac{2}{3}}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \xi^{\frac{2}{3}} \Psi_0''(\overline{Y}) + \dots \tag{2.20}$$

for $\xi \to 0$ with $\overline{Y} = O(1)$ (fixed). In the second region, where $\overline{Y} = O(\xi^{\frac{1}{3}})$, the flow is described by

$$\Psi(\xi, \eta) = -2^{\frac{1}{2}} C + \tfrac{1}{2} a G^{-\frac{1}{2}} \eta^2 \xi^{\frac{2}{3}} + F_1(\eta)\xi + \dots, \quad \xi \to 0, \quad \eta \text{ fixed}, \tag{2.21}$$

where $F_1(\eta)$ is given by (2.19). These solutions are not uniformly valid for $\xi \to 0$. One may infer from (2.20) that, for $\overline{Y} = O(1)$, the vertical velocity v behaves like $\xi^{-\frac{1}{3}} R^{-\frac{1}{2}}$, while the horizontal component u is well behaved and is $O(1)$. In this region, the slope of a streamline is thus proportional to $\xi^{-\frac{1}{3}} R^{-\frac{1}{2}}$, which becomes large as $\xi \to 0$. The slope at the edge of the boundary layer is proportional to an induced pressure in the external flow, which consequently affects the flow in the boundary layer. Hence, as $\xi \to 0$, the interaction pressure gradient P_x is given by

$$P_x = O(\xi^{-\frac{4}{3}} R^{-\frac{1}{2}}),$$

which will become relatively large as $\xi \to 0$. The presence of a large pressure gradient, which has been entirely ignored in the foregoing formulation, suggests the necessity of the local interaction theory.

3. The 'triple-deck' theory

The appropriate form of the triple-deck theory for this problem can be found by arguments similar to those used by Stewartson (1969), Messiter (1970) and Smith (1972, 1973). The essential point to note here is that the disturbance is caused by the $O(R^{-\frac{1}{2}})$ transverse velocity near the wall changing rapidly over a short distance. This relatively weak disturbance (in comparison with those found in shock-wave interactions or near trailing edges for example) leads to a 'linearized' triple-deck analysis with a local pressure variation and main-deck transverse velocity $O(Re^{-\frac{3}{8}})$ rather than the more classical magnitude $O(Re^{-\frac{1}{4}})$. All other scaling parameters are the same as those found in usual triple-deck analyses. The linearized formulation has been used by Smith (1972) for strong-injection initiation studies and recently by Messiter & Hu (1975) to describe boundary-layer variations near a discontinuity in wall curvature.

As in other related local interaction studies, the analysis can be reduced to an equation describing the flow in the lower deck and the related boundary conditions. The latter include a pressure interaction statement arising from the disturbance of the external flow by the effective slender body composed of the internal structure of the triple-deck. Following traditional arguments, one finds that the lower-deck system is given by

$$\tilde{U}_{1\hat{\xi}} + \tilde{V}_{1\hat{Y}} = 0, \tag{3.1}$$

$$(a/T_w^2 G^{\frac{1}{2}})(\hat{Y}\tilde{U}_{1\hat{\xi}} + \tilde{V}_1) = -P_{1\hat{\xi}} + \mu_w \tilde{U}_{1\hat{Y}\hat{Y}}, \tag{3.2}$$

$$\tilde{U}(\hat{\xi}, 0) = 0, \quad \rho_w \tilde{V}(\hat{\xi}, 0) = 2^{\frac{1}{2}} C[1 - H(\hat{\xi})], \tag{3.3}$$

$$\tilde{U}_1(\hat{\xi}, \hat{Y} \to \infty) = \frac{a}{G^{\frac{1}{2}} T_w} \hat{A}(\hat{\xi}) + \frac{Ca}{2^{\frac{3}{2}} G^{\frac{3}{2}} T_w^2} \hat{Y}^2, \tag{3.4}$$

$$\tilde{U}_1(\hat{\xi} \to -\infty, \hat{Y}) = \frac{Ca}{2^{\frac{3}{2}} G^{\frac{3}{2}} T_w^2} \hat{Y}^2, \tag{3.5}$$

$$P_1(\hat{\xi}) = -\hat{A}'(\hat{\xi})/B^{\frac{1}{2}}, \quad B = M_\infty^2 - 1. \tag{3.6}$$

In this set $\hat{\xi} = \xi R^{\frac{3}{8}}$, $\hat{Y} = yR^{\frac{5}{8}}$, $P_1 = PR^{\frac{3}{8}}$ and $\tilde{V}_1 = V$. The velocity parallel to the plate,

$$U = (a/G^{\frac{1}{2}} T_w)\hat{Y}R^{-\frac{1}{8}} + \tilde{U}_1 R^{-\frac{1}{4}} + \ldots, \tag{3.7}$$

defines the perturbation quantity $\tilde{U}_1$. In this expression, the first term represents the basic linear velocity profile in the lower deck. The constants C, a, T_w, μ_w, ρ_w and G have been defined previously. In (3.4) and (3.6) the function $\hat{A}(\hat{\xi})$ represents the slender-body interaction effect. It is found from the complete solution of (3.1)–(3.6). Once the flow in the sublayer is known, and hence $P_1(\xi)$ is determined, the main- and upper-deck solutions can be determined completely.

Although the lower-deck transverse velocity is $O(R^{-\frac{1}{2}})$, it is noted that the main deck value is $O(R^{-\frac{3}{8}})$. This slightly larger value reflects the sudden local downward displacement of steamlines in the main deck as those in the lower deck collapse towards the wall just after injection cut-off. As a result, the outer displacement streamline of the main deck has a local negative slope $O(R^{-\frac{3}{8}})$. This leads to the local pressure interaction $P = O(R^{-\frac{3}{8}})$. Of course, an outer-deck transition zone is required to reduce the pressure disturbance to the globally consistant value $O(R^{-\frac{1}{2}})$.

4. Solution of the sublayer equations

The system of equations (3.1)–(3.6) describing the sublayer flow consists of linear partial differential equations for $\tilde{U}_1$ and $\tilde{V}_1$. It is convenient to reduce these equations to a canonical form independent of the parameters present. This can be achieved through transformations of the type suggested by Stewartson & Williams (1969):

$$\hat{\xi} = \frac{G^{\frac{3}{8}} T_w^{\frac{3}{2}}}{a^{\frac{5}{4}} B^{\frac{3}{8}}} \xi^*, \quad \hat{Y} = \frac{G^{\frac{5}{8}} T_w^{\frac{3}{2}}}{a^{\frac{3}{4}} B^{\frac{1}{8}}} Y^*, \tag{4.1), (4.2}$$

$$\tilde{U}_1 = \frac{C T_w}{2^{\frac{1}{2}} a^{\frac{1}{2}} G^{\frac{1}{4}} B^{\frac{1}{4}}} (U^* + \tfrac{1}{2} Y^{*2}), \quad \tilde{V}_1 = \frac{C T_w V^*}{2^{\frac{1}{2}}}, \tag{4.3), (4.4}$$

$$P_1 = \frac{C T_w^{\frac{1}{2}}}{2^{\frac{1}{2}} a^{\frac{1}{4}} G^{\frac{1}{8}} B^{\frac{3}{8}}} P^*, \quad \hat{A} = \frac{C T_w^2 G^{\frac{1}{4}}}{2^{\frac{1}{2}} a^{\frac{3}{2}} B^{\frac{1}{4}}} A^*. \tag{4.5), (4.6}$$

The form of (4.3) is chosen in order to make the initial condition on U^*, corresponding to (3.5), homogeneous. If we substitute (4.1)–(4.6) into the sublayer equations, we obtain the canonical form

$$U^*_{\xi^*} + V^*_{Y^*} = 0, \tag{4.7}$$

$$Y^* U^*_{\xi^*} + V^* = -P^*_{\xi^*} + U^*_{Y^* Y^*} + 1, \tag{4.8}$$

$$U^*(\xi^*, 0) = 0, \quad V^*(\xi^*, 0) = (1 - H(\xi^*)), \tag{4.9), (4.10}$$

$$U^*(\xi^*, Y^* \to \infty) = A^*(\xi^*), \quad U^*(\xi^* \to -\infty, Y^*) = 0, \tag{4.11), (4.12}$$

$$P^*(\xi^*) = -dA^*(\xi^*)/d\xi^*. \tag{4.13}$$

If the substitutions $v = 1 - V^*$, $u = -U^*$, $r = -P^*$ and $a = -A^*$ are made in (4.7)–(4.13) the system given by Smith & Stewartson [1973*a*, equation (3.2)] results. Hence their equations (3.3)–(3.6) and (3.8) can be used to obtain the pressure interaction and wall shear in the present problem. In our notation it is found that

$$P^*(\xi^*) = \begin{cases} -\frac{3}{4} e^{\theta\xi^*}/\theta, & \xi^* < 0, \quad (4.14) \\ \dfrac{-3^{\frac{1}{2}}}{2\pi\theta} \displaystyle\int_0^\infty \frac{e^{-\theta\xi^* t}\, t^{-\frac{2}{3}}\, dt}{1 + t^{\frac{4}{3}} + t^{\frac{8}{3}}}, & \xi^* > 0, \quad (4.15) \end{cases}$$

$$U^*_{Y^*}(\xi^*, 0) = \begin{cases} \frac{9}{4} \mathrm{Ai}(0)\, e^{\xi\theta^*}/\theta^{\frac{5}{3}}, & \xi^* < 0, \quad (4.16) \\ \dfrac{3^{\frac{3}{2}} \mathrm{Ai}(0)}{2\pi\theta^{\frac{5}{3}}} \left[3\Gamma(\tfrac{2}{3})\, (\xi^*\theta)^{\frac{1}{3}} + \displaystyle\int_0^\infty \frac{t^{\frac{4}{3}}\, e^{-\theta\xi^*}\, dt}{1 + t^{\frac{4}{3}} + t^{\frac{8}{3}}} \right], & \xi > 0, \quad (4.17) \end{cases}$$

where $\theta^{\frac{4}{3}} = -[3\,\mathrm{Ai}'(0)]$ and Ai is the standard Airy function.

In order to plot $P^*(\xi^*)$ and $U^*_{Y^*}(\xi^*, 0)$ for different ξ^*, (4.15) and (4.17) were integrated numerically. A standard Gaussian-type integration subroutine was used. The substitution $u = t^{\frac{1}{3}}$ was made in (4.15) to remove the singularity at

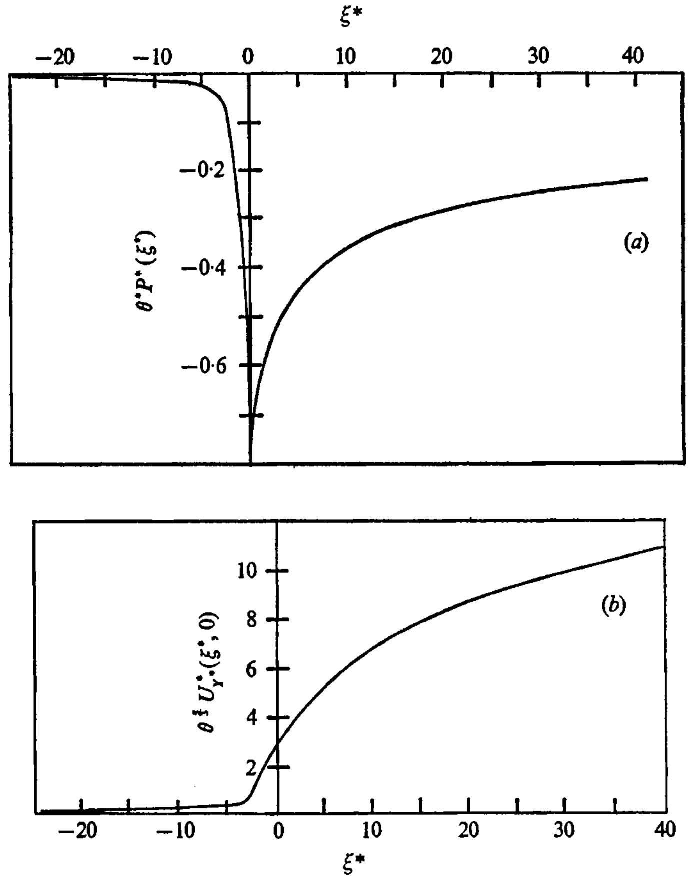

FIGURE 1. The distribution of (a) the pressure $\theta^*P^*(\xi^*)$ and (b) the wall shear $\theta^{\frac{4}{3}}U^*_{Y^*}(\xi^*, 0)$ in the interaction zone.

$t = 0$. For each ξ^* the integrations were carried out over the inverval $0 \leqslant u \leqslant 20$. Owing to the exponential nature of the integrand, replacing the upper limits of the integrals (4.15) and (4.17) by 20 produces at most a transcendentally small error $O(\exp(-u^3))$, $u > 20$. The integrations were stopped at $\xi^* = 40$. For $\xi^* > 40$, asymptotic estimates of (4.15) and (4.17) can be used more effectively. $\theta P^*(\xi^*)$ and $\theta^{\frac{4}{3}}U^*_{Y^*}(\xi^*, 0)$ are plotted *vs.* ξ^* in figures 1 (a) and (b) respectively. The values of $P_1(\bar{\xi})$ and $\tilde{U}_{1\bar{Y}}(\bar{\xi}, 0)$ can be obtained by making use of the transformations (4.1)–(4.6). It is to be noted that these solutions are continuous at $\xi^* = 0$ while the derivatives of $P^*(\xi^*)$ with respect to ξ^* are not. This discontinuity reflects the discontinuity in the boundary conditions at $\xi^* = 0$. The discontinuity in the pressure gradient is, however, much weaker than the original jump in the boundary conditions at $\xi^* = 0$. Further investigation of the problem in the neighbourhood of $\bar{\xi} = 0$ (in contrast to $\xi = 0$) would eliminate such a discontinuity altogether. Finally, it can be shown quite simply (Smith 1972; Smith & Stewartson 1973*a*) that the solution (4.17) actually merges with the downstream solution (2.21).

5. Conclusions

Linearized local interaction theory has been used to describe the rapid transitions occurring in an attached supersonic boundary layer near a point of injection cut-off. The triple-deck analysis provides continuous variation of the dependent variables from the upstream injection boundary layer to the downstream boundary layer described by the Goldstein analysis. In figure 1 (*a*) we observe a strong favourable pressure gradient for $x < 1$ resulting from the sudden downward displacement of streamlines in the boundary layer due to injection cut-off. There is a subsequent more gradual pressure increase for $x > 1$ such that the $O(R^{-\frac{3}{8}})$ pressure field vanishes far from the cut-off point on the triple-deck scale. This is, of course, required by the global requirement that the pressure field be no larger than that of second-order boundary-layer theory, e.g. $O(R^{-\frac{1}{2}})$. It should be noted from (4.5) that the interaction pressure is linearly proportional to the upstream injection-rate constant C.

The wall-stress distribution in the interaction zone may be calculated from the wall shear obtained from (3.7), (4.3) and the results in figure 1 (*b*). In non-dimensional form we find that

$$\tau_w = (R^{-\frac{1}{2}}/T_w)\,[aG^{-\frac{1}{2}} + R^{-\frac{1}{8}}\,b\,U^*_{Y^*}(\xi^*, 0) + \ldots], \tag{5.1}$$

$$b = Ca^{\frac{1}{4}}\,T_w^{\frac{1}{2}}/2^{\frac{1}{2}}\,G^{\frac{7}{8}}\,B^{\frac{1}{8}}.$$

This may be compared with the classical boundary-layer value at $x = 1-$, just upstream of injection cut-off, found from (2.10):

$$\tau_w = (R^{-\frac{1}{2}}/T_w)\,aG^{-\frac{1}{2}}. \tag{5.2}$$

Finally we have the stress distribution in the wall layer just downstream of cut-off found from the Goldstein analysis in (2.13)–(2.19):

$$\tau_w = (R^{-\frac{1}{2}}/T_w)\,[aG^{-\frac{1}{2}} + \xi^{\frac{1}{3}}\,F_1''(0) + O(\xi^{\frac{2}{3}})]. \tag{5.3}$$

We observe that the stress correction in (5.1) arising from the $\tilde{U}_1$ term in (3.7) is $O(R^{-\frac{1}{8}})$ compared with the leading term. From figure 1 (*b*) it may be observed that the correction becomes significant first just upstream of $\xi^* = 0$ as a result of the sudden downstream acceleration of the fluid by the locally strong favourable pressure gradient shown in figure 1 (*a*). Subsequently the correction grows relatively slowly, but relentlessly, owing to viscous effects in the lower deck as the pressure field decays. Hence, far downstream on the triple-deck scale, the correction becomes as large as the leading term in (5.1). This is necessary in order that the value of the total wall shear merges with that of the Goldstein analysis in (5.3).

If this problem had been analysed purely in terms of classical boundary-layer theory, then only the stress distributions (5.2) and (5.3) would be relevant. From the latter one can show that

$$d\tau_w/dx = (R^{-\frac{1}{2}}/3T_w\xi^{\frac{2}{3}})\,[F_1''(0) + O(\xi^{\frac{1}{3}})].$$

Hence there is a singularity in the wall-stress derivative for $\xi \to 0$. This may be compared with the analogous quantity obtained from (5.1):

$$\frac{d\tau_w}{dx} = R^{-\frac{1}{2}}\left[\frac{CB^{\frac{1}{4}}a^{\frac{3}{4}}}{2^{\frac{1}{2}}T_w^2 G^{\frac{5}{4}}}\frac{\partial^2 U^*}{\partial Y^* \partial \xi^*}(\xi^*, 0) + \ldots\right].$$

Since the curve in figure 1 (*b*) is smooth, this distribution is seen to be well-behaved in the interaction zone.

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